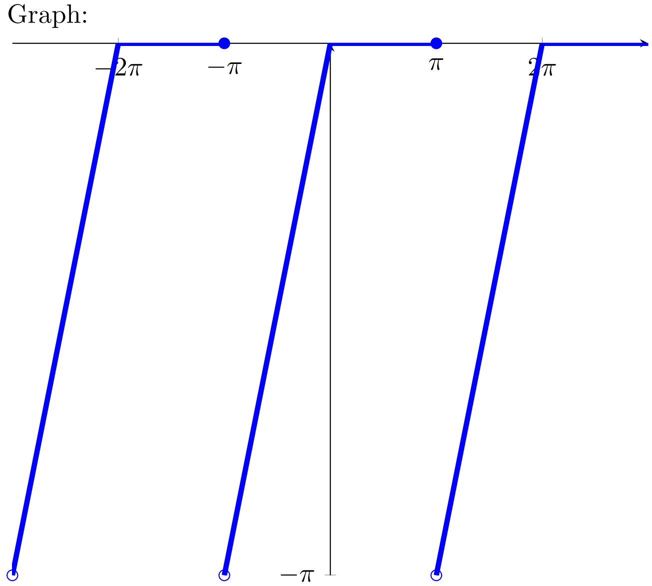
MATH 3060 Assignment 1 solution

Chan Ki Fung

October 9, 2022

1. (a) Graph:



$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(x)dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} xdx \\
&= -\frac{\pi}{4}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\cos nxdx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} x\cos nxdx \\
&= \frac{1}{\pi}\left[\frac{1}{n}x\sin x + \frac{1}{n^2}\cos x\right]_{-\pi}^{0} \\
&= \frac{1}{n^2\pi}(1-(-1)^n)
\end{aligned}
$$

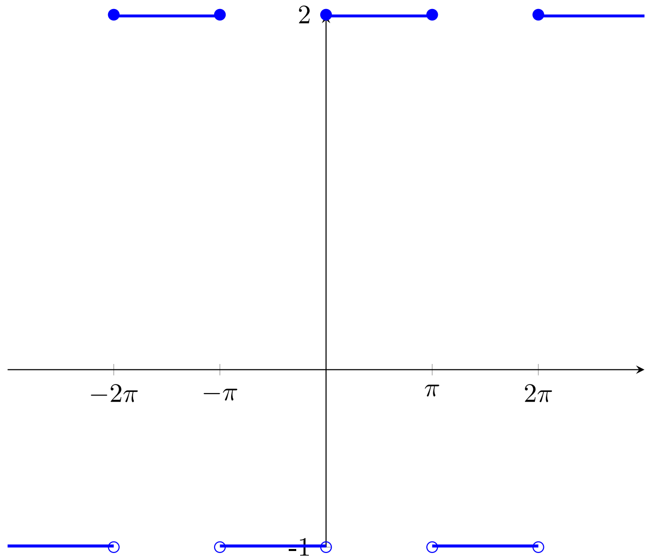
$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(x)\sin nxdx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} x\sin nxdx \\
&= \frac{1}{\pi}\left[-\frac{1}{n}x\cos x + \frac{1}{n^2}\sin x\right]_{-\pi}^{0} \\
&= \frac{1}{n}((-1)^{n+1})
\end{aligned}
$$

Therefore

$$
f_1(x) \sim -\frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.
$$

The Fourier series converges to $-\pi/2$ when x is an odd multiple of π. The fourier series converges to f_1 elsewhere.

(b) Graph:



$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_0^{\pi} 2dx - \frac{1}{2\pi}\int_{-\pi}^{0} dx \\
&= \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{\pi} 2\cos nx dx - \frac{1}{\pi}\int_{-\pi}^{0} \cos nx dx \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} 2\sin nx dx - \frac{1}{\pi}\int_{-\pi}^{0} \sin nx dx \\
&= \frac{3(1-(-1)^n}{n\pi}
\end{aligned}
$$

Therefore

$$
f_2(x) \sim \frac{1}{2} + \frac{6}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin(2n-1)x.
$$

The limit of the Fourier series is the 2π periodic function

$$\begin{cases} 2, & x \in (-\pi, 0) \\ -1, & x \in (0, \pi) \\ \frac{1}{2}, & x \in \{0, \pi\} \end{cases}$$

(c)

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{3x} e^{-inx} dx \\ &= \frac{1}{2\pi} \left[\frac{1}{3 - in} e^{(3-in)x} \right]_{-\pi}^{\pi} \\ &= (-1)^n \frac{e^{3\pi} - e^{-3\pi}}{2\pi(3 - in)} \\ &= \frac{(-1)^n \sinh 3\pi}{(3 - in)\pi} \end{aligned}$$

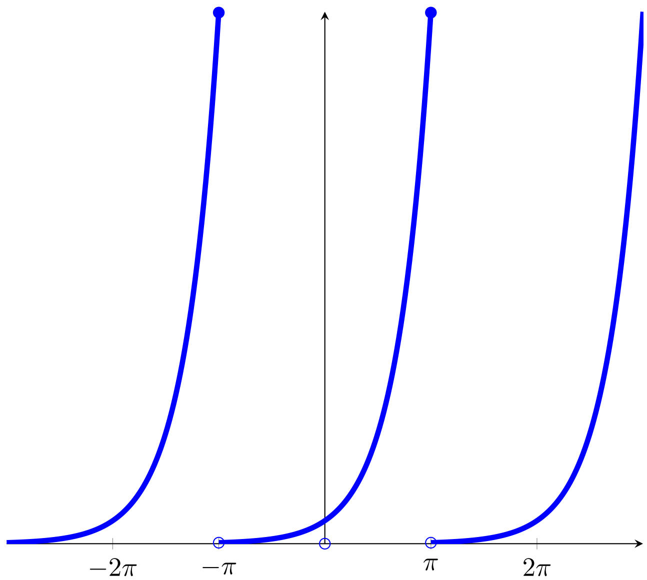
Therefore

$$\begin{aligned} f_3(x) &\sim \frac{\sinh 3\pi}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n}{3 - in} e^{inx} \\ &= \frac{\sinh 3\pi}{3\pi} + \frac{2 \sinh 3\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 9} (3 \cos nx - n \sin nx) \end{aligned}$$

The limit of the Fourier series is the 2π periodic function

$$\begin{cases} e^{3x}, & x \in (-\pi, \pi) \\ \cosh 3\pi, & x = \pi \end{cases}$$

Graph:



(The graph should not be touching the x-axis.)

2. (a) Let $L_1, L_2 > 0$ be such that $|f(x) - f(y)| \leq L_1|x - y|, |g(x) - g(y)| \leq L_2|x - y|$. Also note that since f, g are continuous on $[a, b]$, we can find $M > 0$ so that $|f(x)| \leq M, |g(x)| \leq M$ for $x \in [a, b]$. Now, for $x, y \in [a, b]$, we have

$$|f(x)g(x) - f(y)g(y)| \leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \leq M(L_1 + L_2)|x + y|$$

So gf satisfies the Lipschitz condition with constant $M(L_2 + L_1)$.

(b) Let $L_1, L_2 > 0$ be as in part (a). For any $x, y \in [a, b]$ we have for $x, y \in [a, b]$, we have

$$|g \circ f(x) - g \circ f(y)| \leq L_2|f(x) - f(y)| \leq L_2 L_1|x - y|.$$

So $g \circ f$ satisfies the Lipschitz condition with constant $L_2 L_1$.

3.

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}(x)dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x)dx \\ &= -\pi \end{aligned}$$

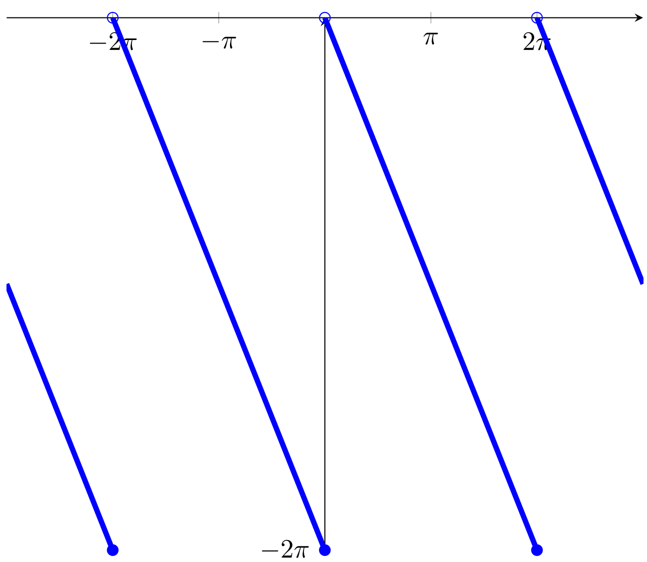
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} -x\cos nx dx \\ &= \frac{1}{\pi}\left[-x\frac{1}{n}\sin nx - \frac{1}{n^2}\cos nx\right]_0^{2\pi} \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} -x\sin nx dx \\ &= \frac{1}{\pi}\left[x\frac{1}{n}\cos nx - \frac{1}{n^2}\sin nx\right]_0^{2\pi} \\ &= \frac{2}{n} \end{aligned}$$

Therefore,

$$f(x) \sim -\pi + 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}.$$

Since f is differentiable with bounded derivative on $(0, 2\pi)$, f is lipschitz continuous at every point in $(0, 2\pi)$, so the Fourier series converges pointwise to f on (0.2π). The limit at 0 is $-\pi$.
Graph:



4. First, $b_n = 0$ for all $n \geq 0$ because f_1 is odd. On the other hand,

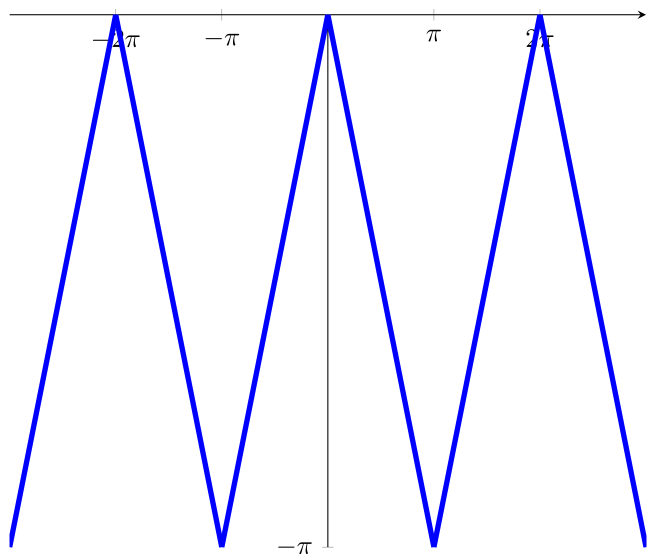
$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{f}_1(x)dx \\ &= \frac{1}{\pi}\int_0^{\pi} f(x)dx \\ &= -\frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} -x\cos nx dx \\ &= \frac{2(1-(-1)^n)}{n^2\pi} \end{aligned}$$

Therefore,

$$\tilde{f}_1(x) \sim -\frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\cos(2n-1)x$$

$\tilde{f}_1$ is piecewise differentiable with bounded derivarives, so it is Lipschitz continuous at every point, so its Fourier series converges to it everywhere. In particular, the limit at 0 is 0. (i.e. $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$)
Graph:


-2π
$-\pi$
π
2π
$-\pi$